Poincaré – Bendixson's Theorem

(& chaos in continuous-dynamic systems)

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The ability of a continuous-dynamic system (CDS) to develop chaotic behavior relates to system dimensionality (dynamic order) and nonlinearity

(We think of system with zero or constant inputs only (so far))

Let's begin with 2^{nd} order dynamic linear CDS:

$$\ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = b_0 u(t) \qquad (1)$$

(constant or zero)

(1) is analogical to it's state-space representation:

$$\dot{x}_1(t) = \alpha_1 x_1(t) + \alpha_2 \{x(t)\} \qquad (2)$$

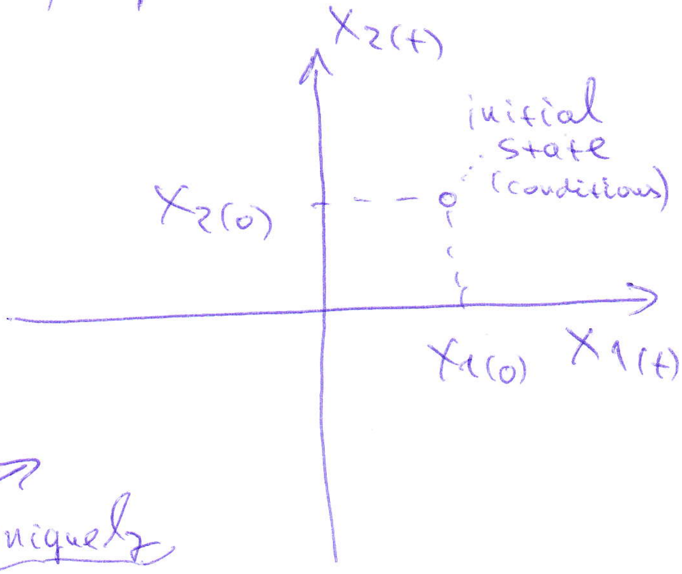
$$\dot{x}_2(t) = \alpha_3 x_1(t) + \alpha_4 \{x(t)\} + \beta$$

$\Leftrightarrow$ 2-D system

There are 2 dynamic variables $x_1(t)$, $x_2(t)$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \Leftarrow \text{State vector}$$

State space



State variables (+ inputs, if any) uniquely determine the system behavior

CDS $\rightarrow$ uniquely calculate $\dot{x}_1(t)$ and $\dot{x}_2(t)$ (i.e. where system is heading to)

discrete system $\rightarrow$ system output in next computation time

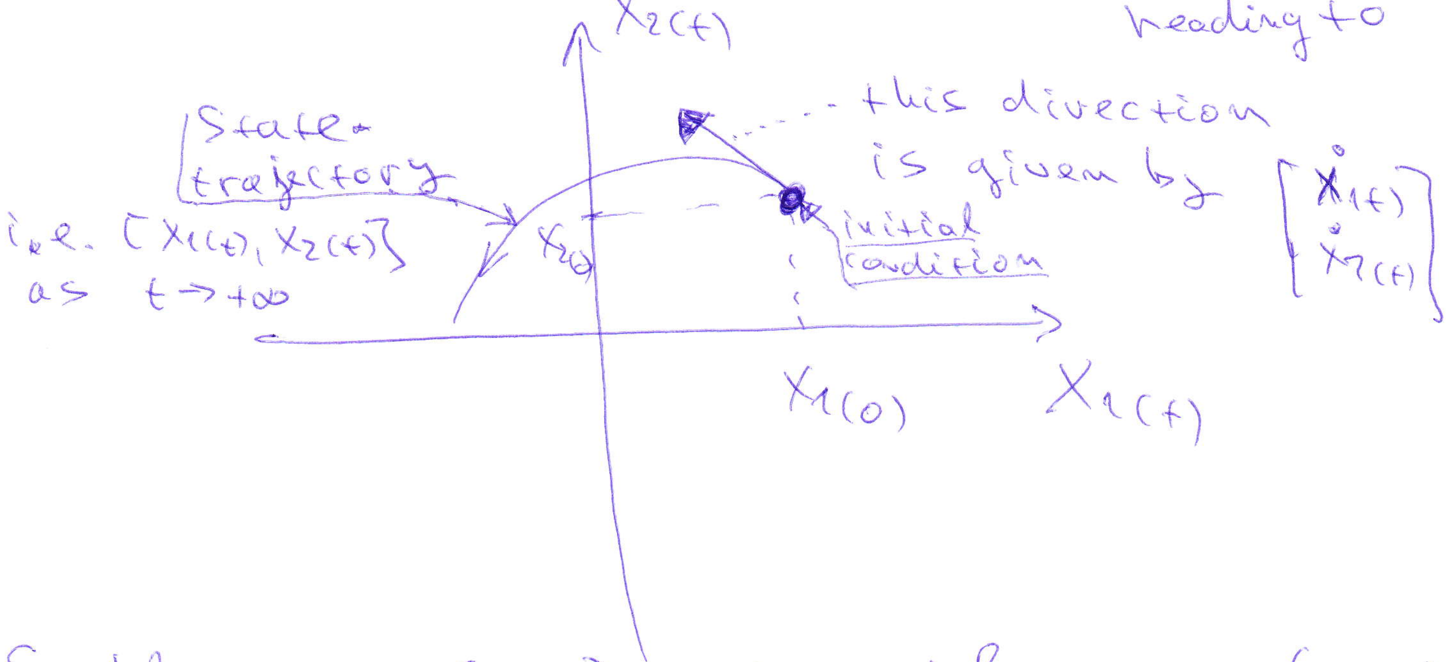
System trajectory in state-space

Recall system (2) from previous page:

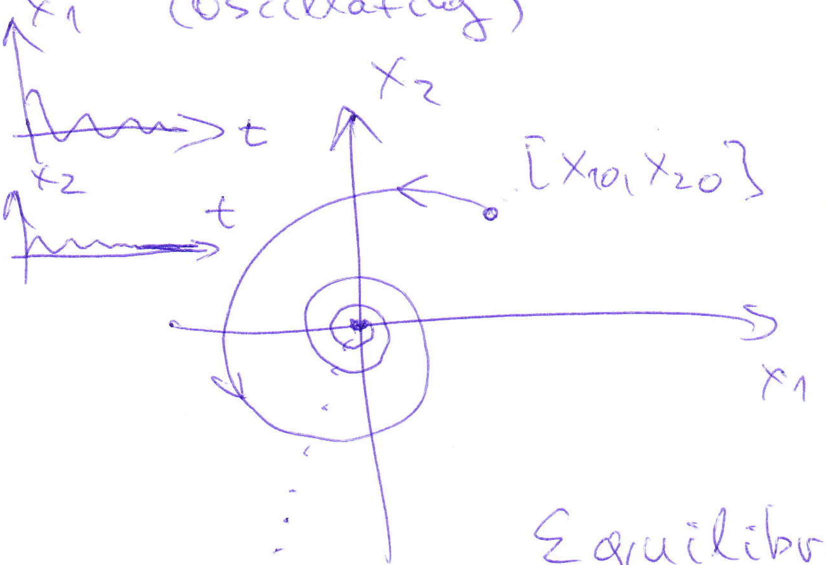
$$\dot{x}_1(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) \quad (2)$$

$$\dot{x}_2(t) = \alpha_3 x_1(t) + \alpha_4 x_2(t) + \beta \;\ldots\; \text{constant or } 0$$

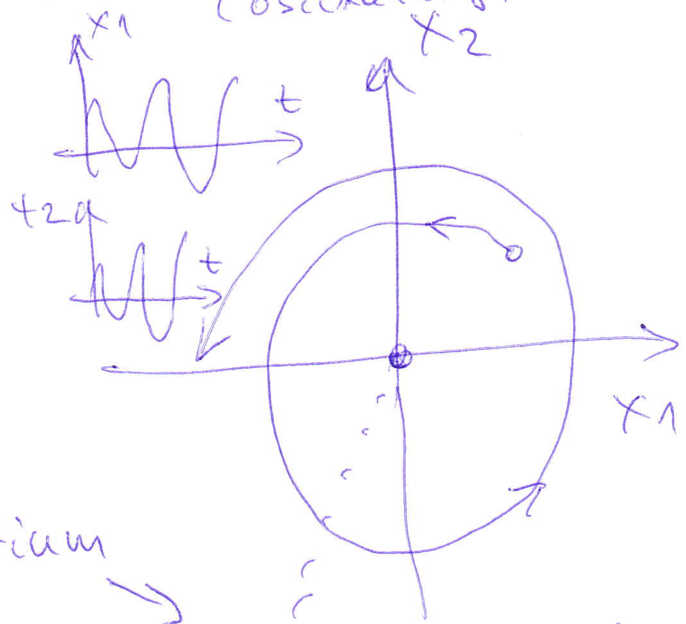
$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ ... state space $\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}$ ... where ~~are~~ is the system heading to



Stable system (CDS)
(Oscillating)



Unstable system (CDS)
(oscillating)



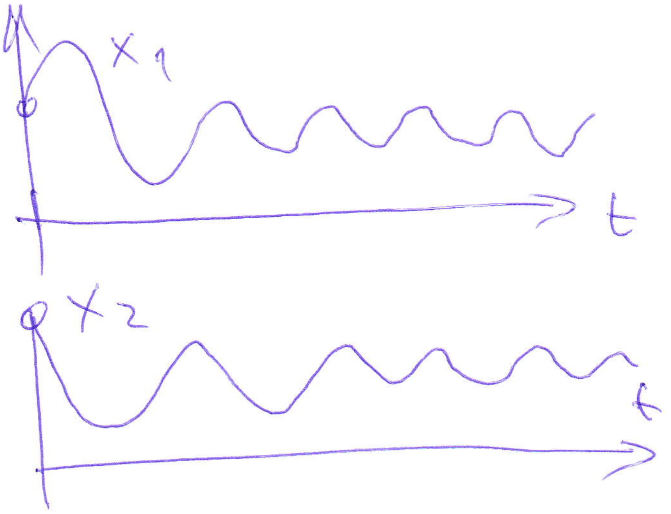
Equilibrium → Sink (Attractor), Limit cycle, Source (Reppeller)

Equilibrium {Sink, Limit cycle, Source}: Limit cycle is a special type of equilibria.

- For linear CDS, limit cycle is also the edge of stability

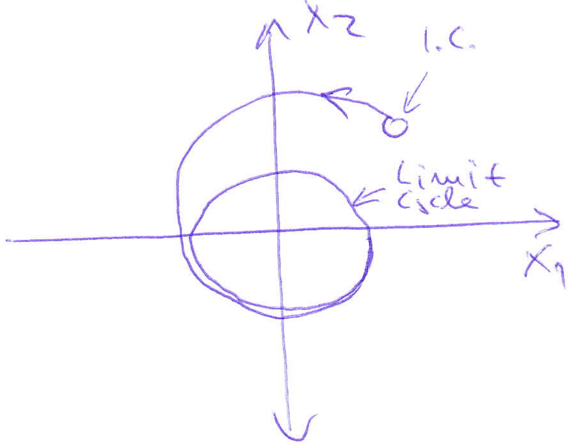
Again, the example of 2-D linear CDS:

Starting "out of" a limit cycle:

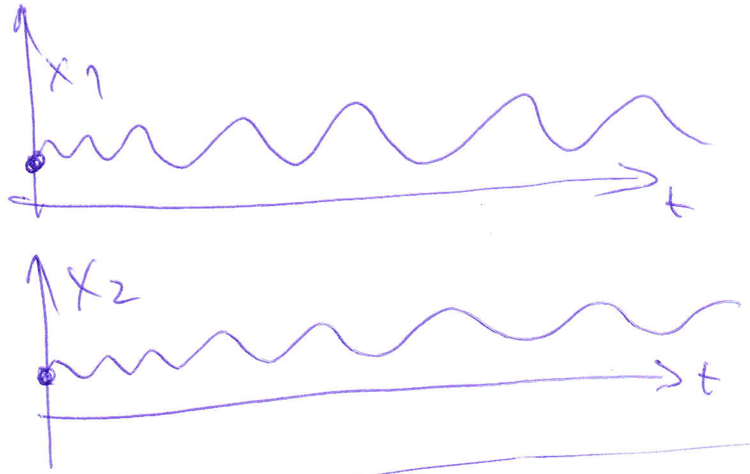


$\Longleftrightarrow$

as state trajectory:

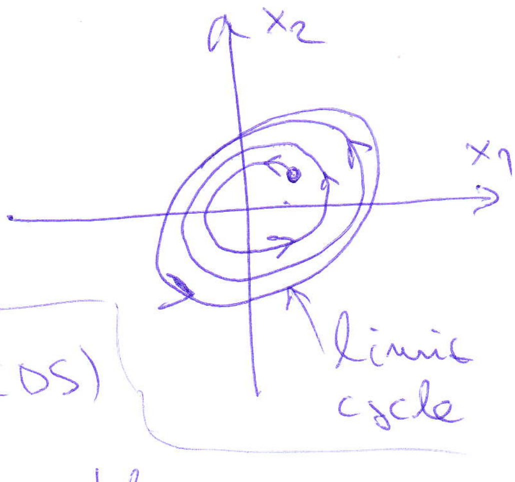


Starting "inside" a limit cycle:



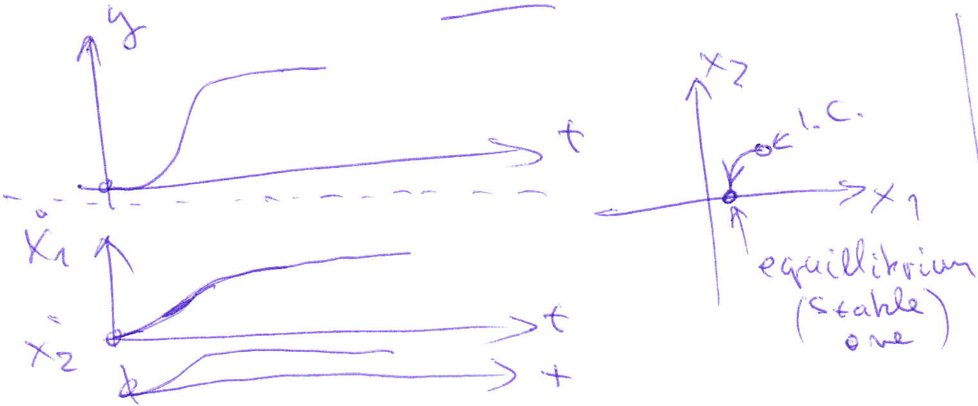
$\Longleftrightarrow$

and in state space: (state trajectory)

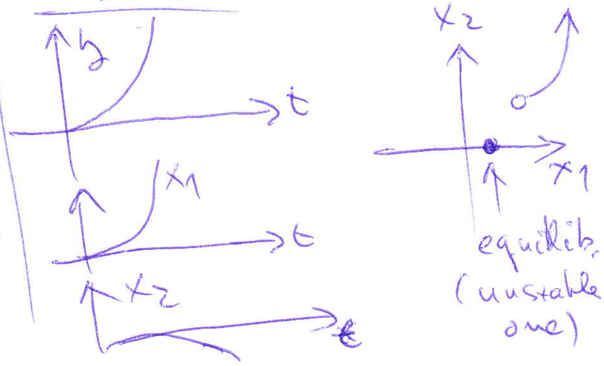


Notice: Non-oscillating system (linear CDS)

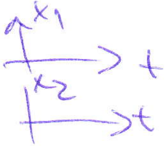
stable



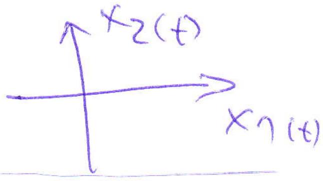
Unstable



The above are plots of 2-D linear CDS behaviors either of single state variabls (individual)



or all of them in N-D $(\dot{2})$ state space (state trajectory) (as the time goes)



Number of equillibrium points (Linear vs. Nonlinear CDS)

How many equilibria can we find for a given system? (Except limit cycles)

Let's look for steady states where $\dot{x} = 0$

$\ddot{x} = 0$

$\vdots$

$$\left.\begin{aligned}\dot{x}_1 = 0 = \alpha_1 \bar{x}_1 + \alpha_2 \bar{x}_2 + \gamma_1 = 0 \\ \dot{x}_2 = 0 = \alpha_3 \bar{x}_1 + \alpha_4 \bar{x}_2 + \beta = 0\end{aligned}\right\} \Rightarrow \text{How many } [\bar{x}_1, \bar{x}_2] \text{ ?}$$

Nonlinear CDS example (2-D i.e. 2nd order)

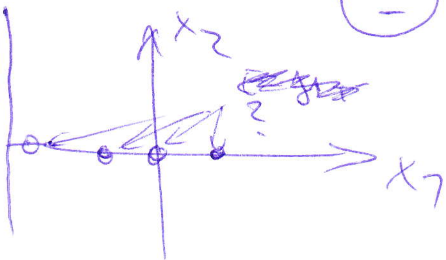
$$\dot{x}_{1(t)} = \alpha_1 x_{1(t)}^2 + \alpha_2 x_{2(t)} + \gamma_1$$

$$\dot{x}_{2(t)} = \alpha_3 x_{1(t)} + \alpha_4 x_{2(t)}^4 + \beta$$

$\Rightarrow$ steady state equations $\Rightarrow$

$$\Rightarrow \left.\begin{aligned}0 = \alpha_1 \bar{x}_1^2 + \alpha_2 \bar{x}_2 + \gamma_1 \\ 0 = \alpha_3 \bar{x}_1 + \alpha_4 \bar{x}_2^4 + \beta\end{aligned}\right\} \Rightarrow \text{How many steady states as } [\bar{x}_1, \bar{x}_2] \text{ ?}$$

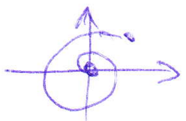
We do not know about the stability, we just found the equilibrium points regardless their stability (sink x source)



Examples of 2nd order (2-D) dynamics system with polynomial nonlinearity of order 1, 2, 3, ...

1) 2nd order dynamics
1st order of nonlinearity
$\Leftrightarrow$ Linear CDS
as in (2) on page 1

Notice: Those are not same:
Order of dynamics
Order of polynomial



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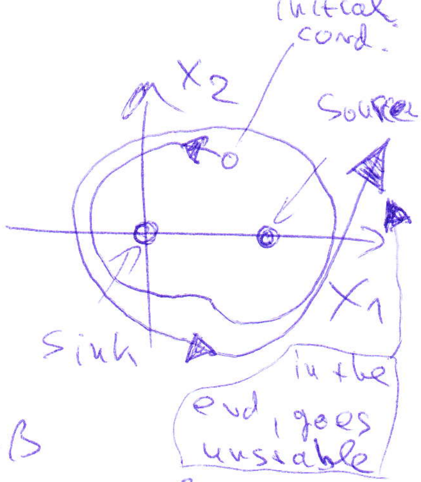
2) 2nd order dynamics
2nd order nonlinearity

e.g.:
$$\dot{x}_1(t) = \alpha_1 x_1(t) + \alpha_2 x_2^2(t)$$
$$\dot{x}_2(t) = \alpha_3 x_1^2(t) + \alpha_4 x_2(t) + \beta$$



Often, there is one stable and one unstable equilibria

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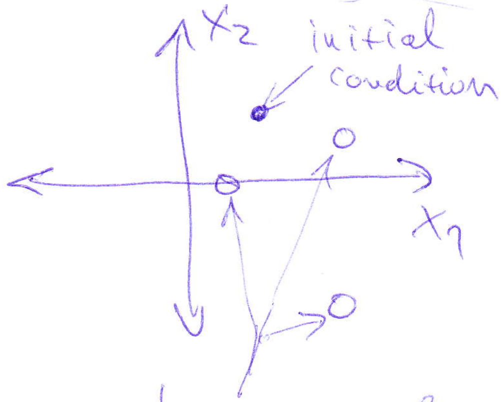
3) 2nd order dynamics
3rd order nonlinearity:



e.g.:
$$x_1(t) = \alpha_1 x_1(t) + \alpha_2 x_2^3(t)$$
$$x_2(t) = \alpha_3 x_1(t) x_2^2(t) + \alpha_4 x_2^2(t) + \beta$$

$\Rightarrow$ In principle, 3 steady state points in state space

Poincaré - Bendixsom's Theorem

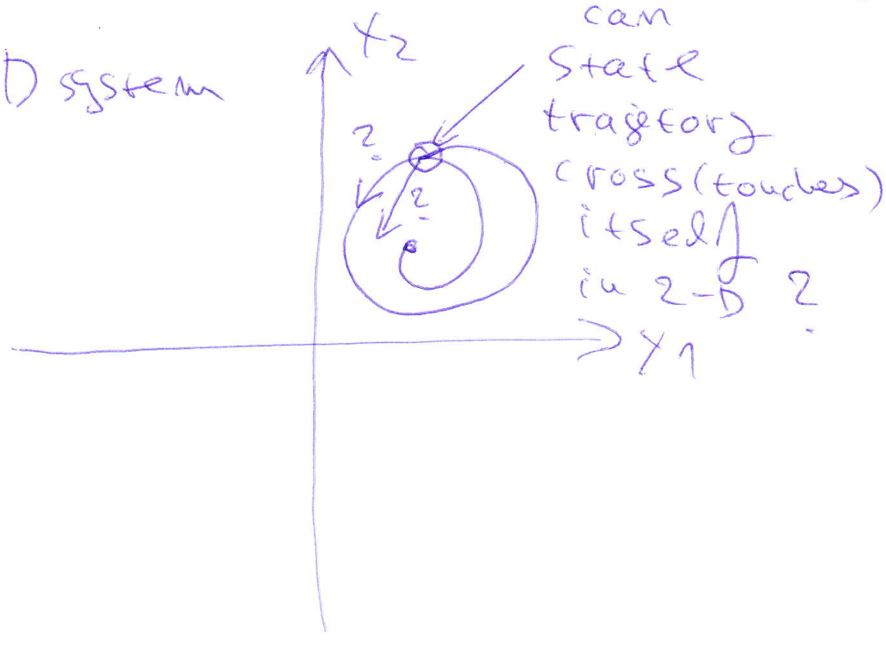
(for continuous-time dynamic system)

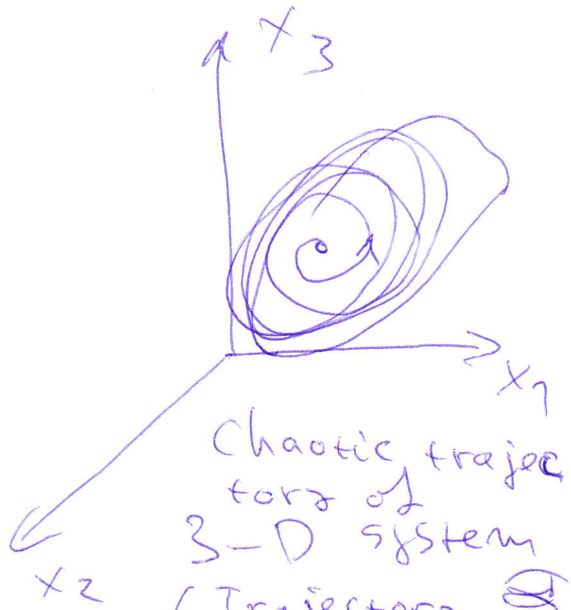
This theorem states that chaotic behavior may not develop with continuous-time dynamic systems in 1-D or 2-D. (constant or zero inputs)

- <u>chaotic behavior</u> = (deterministic chaos); system behavior is impredictable yet it does not become unstable; the system behavior is governed by strictly deterministic equations (no random inputs or varying parameters)

In simple words, we may say that nonlinear dynamic systems have multiple equilibria, each equilibrium may have distinct properties (attracts or reppels a trajectory) thus the system behavior may become very complicated.

2-D system





Chaotic trajectory of 3-D system (Trajectory does not cross itself)